PHYS 216   2008W- T2   Solutions to assignment # 6

Part1:

Assignment 6

1 (a) x_0 = equilibrium position,

$F = -k(x-x_0)$.

$F = -\frac{dU}{dx} \Rightarrow U = \frac{1}{2}kx^2 - kx_0x + C$

$\Rightarrow U = \frac{1}{2}kx^2 - kx_0x + C$

$= \frac{1}{2}k(x-x_0)^2 \underbrace{- \frac{1}{2}kx_0^2 + C}_{= C'}$

Can take $U(x_0) = 0 \Rightarrow C' = 0$,

$\therefore U(x) = \frac{1}{2}k(x-x_0)^2$

Taking $x_0 = 0$, $U(x) = \frac{1}{2}kx^2$.

b) w/o conservation of energy:

Applying Newton's 2nd law to the mass,

$m\frac{d^2x}{dt^2} = -kx$ (taking $x_0 = 0$).

General soln. is $x(t) = A\sin(\omega t + \varphi)$ for some constants, A, φ, and where $\omega = \sqrt{\frac{k}{m}}$.

But $x_0 = x(0) = A\sin\varphi \Rightarrow \varphi = 0$, and

$v_0 = v(0) = \frac{dx}{dt}\Big|_{t=0} = A\omega\cos\varphi = A\omega \Rightarrow A = \frac{v_0}{\omega}$

$\therefore$ The maximum displacement is $A = \frac{v_0}{\omega} = v_0\sqrt{\frac{m}{k}}$.

Part 2

w/ conservation of energy:

$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}$

At the turning points, $v = 0$ and $x = A$, the maximum displacement.

$\Rightarrow E = \frac{1}{2}kA^2$.

At equilibrium, $x = 0$, $v = v_0$.

$\therefore \frac{1}{2}kA^2 = \frac{1}{2}mv_0^2$

$\Rightarrow A = v_0\sqrt{\frac{m}{k}}$ as before.

c) The equation of motion we obtain by the first method is:

$$m\ddot{x} = -kx - \mu mg\frac{\dot{x}}{|\dot{x}|} = -kx - \mu mg \quad \left(\begin{array}{l}\text{before the first}\\ \text{turning point } x = A_1\end{array}\right)$$

$$\Rightarrow x(t) = A\sin(\omega t + \varphi) - \frac{\mu g}{\omega^2} \qquad \left(\omega = \sqrt{\frac{k}{m}}\right)$$

$$0 = x(0) = A\sin\varphi - \frac{\mu g}{\omega^2}, \quad v_0 = \dot{x}(0) = A\omega\cos(\varphi)$$

$$\Rightarrow A = \sqrt{\left(\frac{\mu g}{\omega^2}\right)^2 + \left(\frac{v_0}{\omega}\right)^2}, \quad \therefore A_1 = A - \frac{\mu g}{\omega^2} = \frac{\mu g}{\omega^2}\left(\sqrt{1+\left(\frac{v_0\omega}{\mu g}\right)^2} - 1\right)$$

The second method gives us:

$$-\mu mg A_1 = \int_0^{A_1} -\mu mg\, dx = W_{nc} = E_1 - E_0 = \frac{1}{2}kA_1^2 - \frac{1}{2}mv_0^2$$

$$\Rightarrow A_1 = \frac{-\mu mg + \sqrt{(\mu mg)^2 + kmv_0^2}}{k} = \frac{\mu mg}{k}\left(\sqrt{1 + \frac{kv_0^2}{m(\mu g)^2}} - 1\right)$$

take +ve soln.

same!

2) The moment the pivot axle is removed, there ceases to be any external forces acting on the slab.
$\Rightarrow M\vec{V}_{com} = \text{constant}$ afterward.
At that moment, $\vec{V}_{com} = \vec{\omega} \times \vec{R}_{com} = (-\Omega\hat{k}) \times (d\hat{j})$
$= \Omega d\hat{i}$, where we have noted that $\vec{R}_{com} = d\hat{j}$
for some $0 < d < L$ by symmetry.
The energy of the slab at this moment is (by the parallel axis theorem)
$\frac{1}{2}(I_{com} + Md^2)\Omega^2$ and is conserved $\forall$ later times.

$$\Rightarrow \frac{1}{2}(I_{com} + Md^2)\Omega^2 = \frac{1}{2}I_{com}\omega^2 + \frac{1}{2}Mv_{com}^2$$
$$= \frac{1}{2}I_{com}\omega^2 + \frac{1}{2}M(\Omega d)^2$$

$$\Rightarrow \omega = \Omega$$

Intuitively we know that the direction of $\vec{\omega}$ must not change : $\vec{\omega} = -\Omega\hat{k}$ afterward.

Now,
$$Md\hat{j} = \int_0^L \left((x\hat{i} + y\hat{j})\rho \int_{-y\frac{W}{2L}}^{y\frac{W}{2L}} dx\right) dy = \int_0^L y^2 \frac{\rho W}{L}\hat{j}\, dy = \frac{\rho W L^2}{3}\hat{j}$$

and $M = \frac{1}{2}WL\rho$. $\Rightarrow d = \frac{2L}{3}$

$$\therefore \vec{r}(t) = \frac{2\Omega L}{3}t\,\hat{i} + \frac{2L}{3}\hat{j}, \quad \vec{\omega}(t) = -\Omega\hat{k}$$

Part 4

3a) Energy conservation:

$$mgH = \frac{1}{2}mv^2 + mgh(x) = \frac{1}{2}mv^2 + mgcx^2$$

$$\Rightarrow v = \sqrt{2gH(1 - cx^2/H)}$$

Now, $v^2 = v_x^2 + v_y^2 = \dot{x}^2 + \left(\frac{d}{dt}(cx^2)\right)^2 = \dot{x}^2(1 + (2cx)^2)$

$$\Rightarrow \dot{x} = \boxed{\sqrt{\frac{2gH(1 - cx^2/H)}{(1 + (2cx)^2)}} = \dot{x}(x)}$$

$$\Rightarrow \dot{x}^2(1 + (2cx)^2) = 2gH(1 - cx^2/H) = 2gc(x_{max}^2 - x^2)$$

For $c \ll 1$, we can ignore second order terms in c and higher:

$$\dot{x}^2 = 2gc(x_{max}^2 - x^2)$$

$$\Rightarrow \dot{u} = \sqrt{2gc}(1 - u^2)^{1/2}, \quad \text{where } u = x/x_{max}.$$

$$\Rightarrow \sqrt{2gc}\,t = \int_0^t (1 - u^2)^{-1/2}\dot{u}\,dt = \int_{\underbrace{u(0)}_{= \frac{x(0)}{x_{max}} = \pm 1}}^{u(t)} (1 - u^2)^{-1/2}du$$

$$= \arcsin(u(t)) - \underbrace{\arcsin(\pm 1)}_{= \pm\pi/2}$$

$$\Rightarrow u(t) = \sin(\sqrt{2gc}\,t \pm \pi/2) = \pm\cos(\sqrt{2gc}\,t)$$

$$\Rightarrow x(t) = \pm x_{max}\cos(\sqrt{2gc}\,t)$$

$$\Rightarrow \dot{x}(t) = \mp x_{max}\sqrt{2gc}\sin(\sqrt{2gc}\,t)$$